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The exam answer key of Maths 1 *1:30 mn* : 01/14/2024

Solution 1 *a) We have*

$$f\left(\frac{1}{2}\right)=\frac{4}{5} \text{ and } f(2)=\frac{4}{5}\ldots\ldots\ldots \textbf{\textit{0.5 pt}}$$

b) Since $f\left(\frac{1}{2}\right)=f(2)$ *but* $2\neq\frac{1}{2}$ *then* f *is not injective* ......***1 pt***
2) We have $f'(x)=\frac{-x^2+1}{(x^2+1)^2}$ *and* $\lim_{x\to-\infty}f(x)=\lim_{x\to+\infty}f(x)=0$ *then*

$$\begin{array}{c|ccccccc}
x & -\infty & & -1 & & 1 & & +\infty \\
\hline
f'(x) & & - & & + & & - & \\
\hline
 & 0 & & & & 1 & & \\
f(x) & & \searrow & & \nearrow & & \searrow & \\
 & & & -1 & & & & 0
\end{array} \quad \ldots\ldots \textbf{\textit{1.5 pt}}$$

We remark $g([-1,1])=[-1,1]$ .....***0.5 pt*** *then* g *is surjective* ***0.25 pt***
Partie II *a-The continuity at the point* $x_0=0$, *We have*

$$\lim_{x\to0}f(x)=\lim_{x\to0}\frac{e^x-1}{x}-1=0=f(0)\ldots\ldots \textbf{\textit{0.5 pt}}$$

then f *is continuous at the point* $x_0=0$
The diffentiabilty at the point $x_0=0$, *we have*

$$\lim_{x\to0}\frac{f(x)-f(0)}{x-0}=\lim_{x\to0}\frac{\frac{e^x-1}{x}-1}{x}=\lim_{x\to0}\frac{e^x-1}{x^2}-\frac{1}{x}=\lim_{x\to0}\frac{e^x-1-x}{x^2}=\frac{0}{0} \qquad I.F.\ldots\ldots \textbf{\textit{1 pt}}$$

we use the hospital rule we have $\lim_{x\to0}\frac{e^x-1-x}{x^2}=\lim_{x\to0}\frac{e^x-1}{2x}=\frac{1}{2}$......***0.5 pt***
b- we have $x\mapsto\frac{e^x-1}{x}-1$ *is diffentiable on* $]-\infty,0[$ *and* $]0,+\infty[$ *because it is a composite diffentiable functions on* $]-\infty,0[$ *and* $]0,+\infty[$, *and it is differentiable at the point* $x_0=0$, *hence* f *is differentiable on* $\mathbb{R}$.....***0,5 pt***
c- The function $x\longmapsto\frac{e^x-1}{x}$ *is continuous for all values of* $x\in]0;1]$ *and note that*

$$\lim_{x\to0}\frac{e^x-1}{x}=1 \text{ and } f(1)=e-1\ldots. \textbf{\textit{0,5 pt}}$$

and 1.5 *is between* 1 *and* $e-1$
Using the Intermediate Value Theorem, the equation $\frac{e^x-1}{x}=1.5$ *have at least a solution in the interval* $]0;1[$....***0,5 pt***

Solution 2 ***Partie I***
1) Let us show that U *is a vector subspace of* $\mathbb{R}^3$
We have
$0_{\mathbb{R}^3}=(0,0,0)\in U$ *since* $\frac{0+0}{2}=0$ *is verify then* $U\neq\varnothing$.......***0.25pt***
Let $u(x,y,z)$ *and* $v(x',y',z')\in U$: *Show that montrons que* $u+v\in U$?

we have

$$\begin{aligned} u\,(x,y,z) &\in U \quad then \quad \frac{x+y}{2}=z \;.....(1)\;....\boldsymbol{0.25pt} \\ and \;\; v\,(x,y,z) &\in U \quad then \quad \frac{x'+y'}{2}=z' \quad .....(2) \end{aligned}$$

We have $u+v\,(x+x',y+y',z+z')\in U,$ *since*

$$\frac{x+x'}{2}+\frac{y+y'}{2}=z+z'....\boldsymbol{0.5pt}$$

then $u+v\in U$

Let $u\,(x,y,z)\in U$ *and* $\alpha\in\mathbb{R}$: *Show that* $\alpha u\in U$?
we have

$$\begin{aligned} u\,(x,y,z) &\in U \;\; then \quad \frac{x+y}{2}=z \;.....(1)\;....\boldsymbol{0.25\ pt} \\ then\,'\;\; \frac{\alpha x+\alpha y}{2} &= \alpha z.....(2)\;...\boldsymbol{0.5\ pt} \end{aligned}$$

then $\alpha u\in U$
Finaly U *is a vector subspace of* $\mathbb{R}^3$
2) The basis of U,

$$\begin{aligned} U &= \{(x,y,z)\in\mathbb{R}^3, \frac{x+y}{2}=z \quad \} \\ &= \{(x,y,z)\in\mathbb{R}^3, x=2z-y \quad \} \\ &= \{(2z-y,y,z)/y,z\in\mathbb{R}\}=\{(-y,y,0)+(2z,0,z)/y,z\in\mathbb{R}\}....\boldsymbol{0.25\ pt} \\ &= \{y(-1,1,0)+z(2,0,1)/y,z\in\mathbb{R}\}....\boldsymbol{0.25pt} \end{aligned}$$

the set $\{u(-1,1,0),v(2,0,1)\}$ *is a basis* U. *Show that* $\{u,v\}$ *is independent*
Let $\lambda_1,\lambda_2\in\mathbb{R}$ *We have*

$$\lambda_1 u+\lambda_2 v=0_{\mathbb{R}^3}\Longrightarrow\left\{\begin{array}{c} -\lambda_1+2\lambda_2=0 \quad .....(1) \\ \lambda_1+0.\lambda_2=0 \;.....(2) \\ 0.\lambda_1+\lambda_2=0 \quad ....(3) \end{array}\right. \Longrightarrow \lambda_1=0 \quad and \quad \lambda_2=0....\boldsymbol{0.5\ pt}$$

then $\{u,v\}$ *is abasis of* U
Partie II
Let us show that f *is a linear map*

Let $u(x,y), v(x',y') \in \mathbb{R}^2$ *We have*

$$\begin{aligned}
f(u+v) &= f(x+x', y+y') = \left(\frac{2}{5}(x+x') - \frac{1}{3}(y+y'), x+x'+\frac{1}{3}(y+y')\right) \ldots\textbf{\textit{0.25pt}} \\
&= \left(\frac{2}{5}x + \frac{2}{5}x' - \frac{1}{3}y - \frac{1}{3}y', x+x'+\frac{1}{3}y+\frac{1}{3}y'\right) \\
&= \left(\frac{2}{5}x - \frac{1}{3}y + \frac{2}{5}x' - \frac{1}{3}y', x+\frac{1}{3}y + x' + \frac{1}{3}y'\right) \\
&= \left(\frac{2}{5}x - \frac{1}{3}y, x+\frac{1}{3}y\right) + \left(\frac{2}{5}x' - \frac{1}{3}y', x'+\frac{1}{3}y'\right) \\
&= f(x,y) + f(x',y') \\
&= f(u) + f(v) \ldots\textbf{\textit{0.25pt}}
\end{aligned}$$

Let $\alpha \in \mathbb{R}$, $u(x,y) \in \mathbb{R}^2$ *We have*

$$\begin{aligned}
f(\alpha u) &= f(\alpha x, \alpha y) = \left(\frac{2}{5}\alpha x - \frac{1}{3}y, \alpha x + -\frac{1}{3}\alpha y\right) \ldots\textbf{\textit{0.25pt}} \\
&= \alpha\left(\frac{2}{5}x - \frac{1}{3}y, x+\frac{1}{3}y\right) \\
&= \alpha f(x,y) \ldots\textbf{\textit{0.25pt}} \\
&= \alpha f(u)
\end{aligned}$$

then f *is a linear map*
2) a) By definition of Kernel of f

$$\begin{aligned}
\ker f &= \left\{(x,y) \in \mathbb{R}^2 / f(x,y) = 0_{\mathbb{R}^2}\right\} \ldots\textbf{\textit{0.25pt}} \\
&= \left\{(x,y) \in \mathbb{R}^2 / \left(\frac{2}{5}x - \frac{1}{3}y, x+\frac{1}{3}y\right) = (0,0)\right\} \ldots\textbf{\textit{0.25pt}}
\end{aligned}$$

then

$$\begin{cases} \frac{2}{5}x - \frac{1}{3}y = 0 \ \ldots\ldots (1) \\ x + \frac{1}{3}y = 0 \ldots\ldots (2) \end{cases} \Rightarrow (1)+(2) \Rightarrow x = 0 \quad et \quad y = 0 \ldots\textbf{\textit{0.25pt}}$$

then

$$\ker f = \{(0,0) = 0_{\mathbb{R}^2}\} \ldots\textbf{\textit{0.25pt}}$$

b) since $\ker f = \{0_{\mathbb{R}^2}\}$ *then* g *is injective*....**_0.25pt_**

We have according to the dimension theorem

$\dim\ker f + \dim \operatorname{Im} f = \dim \mathbb{R}^2 \Rightarrow \dim \operatorname{Im} f = \dim \mathbb{R}^2$ *since* $\dim\ker f = 0$....**_0.5pt_**

f *is surjective*....**_0.25pt_**

Solution 3 *We have*

$$\forall x, y \in \mathbb{R}, x\Re y \Longleftrightarrow \cos^2 x + \sin^2 y = 1$$

1) *Let us show that* $\Re$ *is an equivelente relation*
$\Re$ *is reflexive indeed, for all* $x \in \mathbb{R}$, *we have*

$$x\Re x \Longleftrightarrow \cos^2 x + \sin^2 x \quad \textit{is always true....}\textbf{\textit{0.5pt}}$$

$\Re$ *is symmetricre, indeed, for all* $x, y \in \mathbb{R}$, *we have*

$$x\Re y \Longleftrightarrow x\Re x \Longleftrightarrow \cos^2 x + \sin^2 y = 1....\textbf{\textit{0.5pt}}$$

then remplcing $\cos^2 x$ *by* $1 - \sin x^2$ *and* $\sin^2 y$ *by* $1 - \cos y^2$

$$1 - \sin^2 x + 1 - \cos^2 y = 1...\textit{the proof is complet.}\textbf{\textit{0.5pt}}$$

$\Re$ *est transitive car ,on a pour tout* x, y *et* $z \in \mathbb{R}$,

if $\quad x\Re y \Longleftrightarrow \cos^2 x + \sin^2 y = 1........(1)$ *and* $\quad y\Re z \Longleftrightarrow \cos^2 y + \sin^2 z = 1.......\ (2)$

additionel (1) *and* (2)

$$\cos^2 x + \sin^2 y + \cos^2 y + \sin^2 z = 2....\textbf{\textit{0.5pt}}$$

then

$$\cos^2 x + 1 + \sin^2 z = 2$$

the proof is complet, then $x\Re z$*....**0.5pt***

Since $\Re$ *is symmetric, reflexive and transitive then* $\Re$ *it is an eauivalence relation.....**0.5 pt***

2) $\dot{\frac{\pi}{2}}$ *class equivalence of* $\frac{\pi}{2} \in \mathbb{R}$

$$\begin{aligned} \dot{\frac{\pi}{2}} &= \left\{x \in \mathbb{R}/x\Re\frac{\pi}{2}\right\} = \left\{x \in \mathbb{R}/\cos^2 x + \sin^2 \frac{\pi}{2} = 1\right\}....\textbf{\textit{0.5pt}} \\ &= \left\{x = \frac{\pi}{2} + k\pi/k \in \mathbb{Z}\right\}....\textbf{\textit{0.5 pt}} \end{aligned}$$

Solution 4 *1- The kernel* $\ker f$ *is a subspace of* U *because*

$$u, v \in \ker f \textit{ and show that } u + v \in \ker f$$

we have

$$u \in \ker f \textit{ then } f(u) = 0_V \textit{ and } u \in \ker f \textit{ then } f(v) = 0_V...\textbf{\textit{0.5pt}}$$

since f *is linear we have*

$$f(u + v) = f(u) + f(v) = 0_v...\textbf{\textit{0.5pt}}$$

then $u + v \in \ker f$

and symelary

$$u \in \ker f \text{ and } \alpha \in \mathbb{k} \text{ we ave } \alpha u \in \ker f \ldots\textbf{\textit{0.5pt}}$$

indeed f *is linear we have*

$$f(\alpha u) = \alpha f(u) = 0_v \ldots\textbf{\textit{0.5pt}}$$

2- Let us show that if g *and* h *are linear then* gof *is linear*
Let $u, v \in V$. *as soon as* f *and* g *are linear then*

$$\begin{aligned} (gof)(u+v) &= g(f(u+v)) = gf((u) + f(v)) = \\ &= g(f(u)) + g(f(v)) = (gof)(u) + (gof)(v) \ldots .\textbf{\textit{1pt}} \end{aligned}$$

and for $u \in V$ $,\lambda \in \mathbb{k}$, *since* f *and* g *are linears, we have*

$$\begin{aligned} (gof)(\lambda u) &= g(f(\lambda u)) = g(\lambda f(u)) = \\ &= \lambda g(f(u)) = \lambda (gof)(u) \ldots .\textbf{\textit{0.5pt}} \end{aligned}$$

3- f *is surjective i;e* $f(U) = V$ *and* g *is surjective* $g(V) = W$ *...**0.5pt***
Let us show that gof *is surjective*

we have

$$gof(U) = g(v) = W \ldots\textbf{\textit{1pt}}$$

then gof *is surjective.*